Math 254B Lecture 28 Notes

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1 Furstenberg's Slicing Theorem

1.1 Full construction of good CP distributions

Recap: We have $\Phi_i x = rUx + a_i$, where U is rotation by 2π. This has attractor K with map S. If $z \in \operatorname{supp}(\nu)$, then

$$T(z, \nu) = (Sz, S_* \nu|_{[z]_1}) = (Sz, \nu^{\alpha_1(z)}), \qquad T^t(z, \nu) = (S^t z, \nu^{\alpha_{[1;t]}(z)}).$$

Coming construction:

- Take $\nu^{(0)} \in P(K)$, and define $\widehat{\mu}^{(0)} := \nu^{(0)} \times \delta_{\nu^{(0)}}$.
- Let $\widehat{\mu}^{(n)} = \frac{1}{n} \sum_{t=0}^{n-1} T_*^t \widehat{\mu}^{(0)}$ for each $n \geq 1$.
- Let $\widehat{\mu} := \lim_i \widehat{\mu}^{(n_i)}$ for some weak* convergent subsequence.
- Replace $\widehat{\mu}$ with a "typical" ergodic component.

Recall from last time the function $F(z, \nu) = -\log(\nu([z]_1))$.

Lemma 1.1. *With $\widehat{\mu}^{(0)}$ as above, we have*

$$\int F \, d(T_*^t \widehat{\mu}^{(0)}) = H_{\nu^{(0)}}(\alpha_{t+1} \mid \alpha_1, \dots, \alpha_t).$$

Proof. The left hand side is

$$\begin{aligned}
\int -\log(\nu'([z']_1)) \, d(T_*^t \widehat{\mu}^{(0)})(z', \nu') &= \int -\log(\nu'|_{[z']_1^t}(\underbrace{S^{-t}([S^t z']_1)}_{[z']_{t+1}})) \, d\widehat{\mu}^{(0)}(z', \nu') \\
&= \int -\log(\nu'|_{[z']_1^t}([z']_{t+1}) \, d\nu^{(0)}(z') \\
&= \sum_w \nu^{(0)}(K_w) \int (-\log(\nu^{(0)}|_{K_w}([z']_{t+1})) \, d\nu^{(0)}|_{K_w}(z') \\
&= H_{\nu^{(0)}}(\alpha_{t+1} \mid \alpha_1, \dots, \alpha_t).
\end{aligned}$$

□

Corollary 1.1.

$$\int F\, d\widehat{\mu}^{(n)} = \frac{1}{n} H_{\nu}^{(0)}(\alpha_1, \dots, \alpha_n).$$

Proof. Use the chain rule. □

We want to make sure that when we take our weak* limit, we keep this right hand side large.

Lemma 1.2. *Assume that $\nu^{(0)}(E) \le c(\operatorname{diam}(E))^\alpha$ for all $E \subseteq \mathbb{R}^2$. Then*

$$\int F\, d\mu^{(n)} \ge \alpha \log(r^{-1}) - o(1).$$

Proof. If $w \in [k]^n$, then $\operatorname{diam}(K_w) \subseteq Dr^n$. Then $\nu^{(0)}(K_w) \le cD^\alpha r^{n\alpha}$. Then

$$-\log(\nu^{(0)}(K_w)) \ge -\underbrace{\log(cD^\alpha)}_{O(1)} + n\alpha \log(r^{-1})$$

So $H_{\nu^{(0)}}(\alpha_1, \dots, \alpha_n)$, the average of the left hand side, is $\ge -O(1) + n\alpha \log(r^{-1})$. □

So after the weak* limit,

$$\int F\, d\widehat{\mu} \ge \alpha \log(r^{-1}).$$

Using the ergodic decomposition of $\widehat{\mu}$,

$$\iint F\, d\widehat{\mu}_x d\widehat{\mu}(x) \ge \alpha \log(r^{-1}).$$

1.2 Measures supported on slices

If we want to work with CP-systems $K \times P(K)$ and talk about lines, we should talk about the **CP-angle systems**, $K \times P(K) \times \mathbb{T}$, where $\mathbb{T}$ says which direction the line is in. If $z \in \mathbb{R}^2$ and $\theta \in \mathbb{T}$, then let $L_{z,\theta}$ be the line through z in direction $e^{2\pi i\theta}$. Let

$$\tilde{X} = \{(z, \nu, \theta) \in K \times P(K) \times \mathbb{T} : \nu(K \cap L_{z,\theta}) = 1\}.$$

Lemma 1.3. *$\tilde{X}$ is invariant under $T \times R_{-\xi}$.*

Proof. Let $z \in \operatorname{supp}(\nu)$. Suppose that $\nu(K \cap L_{z,\theta}) = 1$. Then $\nu|_{[z]_1}(K \cap L_{z,\theta}) = 1$. Now

$$\nu^{\alpha_1(z)}(K \cap L_{Sz,\theta-\xi}) = \nu|_{[z]_1}(S^{-1}(K \cap L_{Sz,\theta-\xi})) \ge \nu|_{[z]_1}(K \cap L_{z,\theta}) = 1.$$ □

Lemma 1.4. *$\{\tilde{\mu} \in P(K \times P(K) \times \mathbb{T}) : \widehat{\mu}$ is adapted, $\tilde{\mu}(\tilde{X}) = 1\}$ is weak* closed and convex.*

This set of distributions equals

$$\bigcap_{f\in C(K\times P(K))}\left\{\int f(z,\nu)\,d\tilde{\mu}(z,\nu,\theta)=\int Qf(z,\nu)\,d\widehat{\mu}(z,\nu,\theta)\right\}$$
$$\cap\left\{\int \nu(K\cap L_{z,\theta})\,d\tilde{\mu}(z,\nu,\theta)=1\right\}.$$

Proposition 1.1. *Fix any line L with $\dim(K\cap L)>0$. Then there exists an ergodic $(T\times R_{-\xi})$-invariant, adapted distribution $\tilde{\mu}$ on $K\times P(K)\times\mathbb{T}$ such that $\tilde{\mu}$-a.e. triple (z,ν,θ) lies in*

$$Z=\{(z,\nu,\theta):\nu(L\cap L_{z,\theta})=1,\dim(\nu)\geq\dim(K\cap L).$$

Proof. Let $\alpha:=\dim(K\cap L)$, and assume that $m_\alpha(K\cap L)>0$. Then Frostman's lemma gives $\nu^{(0)}\in P(K\cap L)$ such that $\nu^{(0)}(E)\leq c(\mathrm{diam}(E))^\alpha$ for all E. Let $\theta^{(0)}\in\mathbb{T}$ be such that L is parallel to $e^{2\pi I\theta^{(0)}}$. Let $\tilde{\mu}^{(0)}=\nu^{(0)}\times\delta_{\nu^{(0)}}\times\delta_{\theta^{(0)}}$, let

$$\tilde{\mu}^{(n)}=\frac{1}{n}\sum_{t=0}^{n-1}(T\times R_{-\xi})^t_*\widehat{\mu}^{(0)},$$

and let

$$\tilde{\mu}:=\lim_i\tilde{\mu}^{(n_i)}$$

for some weak* convergent subsequence. This is adapted, $(T\times R_\xi)$-invariant, and

$$\int F(z,\nu)\,d\tilde{\mu}(z,\nu,\theta)\geq\alpha\log(r^{-1}).$$

If we have the ergodic decomposition $\widehat{\mu}=\int\widehat{\mu}_x\,d\widehat{\mu}(x)$ for $(T\times R_{-\xi})$, then there is a $\tilde{\mu}$-positive measure set of x such that $\tilde{\mu}_x$ is adapted, $(T\times R_{-\xi})$-invariant and $\int F\,\tilde{\mu}_x\geq\alpha\log(r^{-1})$. So $\widehat{\mu}_x$ works.

If $m_\alpha(K\cap L)=0$, extra care is needed. We have to let α tend to $\dim(K\cap L)$, instead. □

We can finally prove Furstenberg's theorem:

Proof. By the proposition, product $\tilde{\mu}$ living on Z. Consider the coordinate projection $\varphi:K\times P(K)\times\mathbb{T}\to\mathbb{T}$. The measure $\varphi_*\tilde{\mu}$ is R_ξ-invariant, so it must be Lebesgue measure. So for m-a.e. θ, there exists z,ν such that $(z,\nu,\theta)\in Z$. So $\nu(K\cap L_{z,\theta})=1$, and $\dim(\nu)\geq\alpha$. This gives us that

$$\dim(K\cap L_{z,\theta})\geq\alpha.$$ □